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Bounding Heavy Meson Form Factors Using Inclusive Sum Rules

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Abstract

We utilize inclusive sum rules to construct both upper and lower bounds on the form factors for $\overline{B} \rightarrow D, D^*, \rho, \pi, \omega, K$ and K^* semi-leptonic and radiative decays. We include the leading nonperturbative $1/E$ corrections and point out cases when α_s corrections are equally important. We compute the α_s correction to the lower bound on the $\overline{B} \rightarrow D^*$ form factor $f(w)$ at zero recoil, thereby constraining its normalization $f(1)$ to within $6 - 8\%$ of the upper bound. We show that the $\overline{B} \rightarrow \rho$ form factor a_+ is suppressed at small momentum transfer by either a factor of $1/E$ or α_s . These bounds can be used to rule out phenomenological models as well as to determine values for the CKM matrix elements once radiative corrections are included.

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1 Introduction

While there has been much progress in calculating inclusive decay rates [1-6] of heavy mesons, exclusive rates have still not been tamed within the confines of a first principles calculation. Consequently, the phenomenology of exclusive decays has been relegated to the realm of models which, while quite useful on the qualitative level, leave much to be desired when it comes to quantitative issues. For instance, the CKM matrix element V_{ub} is still only known to within a factor of two, because present extractions are based on model considerations. Inclusive techniques are plagued by large corrections in the theoretical calculations [7], and thus it seems that we have no recourse but to try to tame the exclusive rate. Given that at present, we cannot calculate the form factors themselves, we do the next best thing, which is to *bound* them.

In reference [8], the equivalence of hadronic and partonic expressions for inclusive decay rates was used to derive sum rules analogous to those developed for deep inelastic scattering. These sum rules apply to heavy-heavy as well as heavy-light quark transitions, as long as the energy of the final state hadron is large compared to the QCD scale. An explicit upper bound on the $\overline{B} \rightarrow D^* l \overline{\nu}$ matrix element at zero recoil was presented (although radiative corrections significantly weaken this bound [9]) in [8]. In this paper, we use these inclusive sum rules to compute explicit bounds on individual heavy-heavy form factors at arbitrary momentum transfer, and heavy-light form factors at sufficiently small momentum transfer. In particular we bound form factors for the transitions $\overline{B} \rightarrow H l \overline{\nu}$, where H can be a D, D^*, ρ, ω , or π meson, and $\overline{B} \rightarrow H l \overline{l}$ (or $\overline{B} \rightarrow H \gamma$), where H can be a K^* or K meson. We show how to compute not only upper bounds, but *lower* bounds as well, and present the explicit bounding functions. Phenomenological issues like the extraction of V_{ub} will be addressed in a subsequent publication, since such analyses require the inclusion of possibly large radiative corrections that are not included in the present article.

2 Constructing Sum Rules

Consider the semi-leptonic decay of a B meson to a hadron H through a vector $V^\mu = \overline{q}\gamma^\mu b$ or axial $A^\mu = \overline{q}\gamma^\mu\gamma_5 b$ flavor-changing current. Quark-hadron duality permits us to reliably calculate the inclusive rate, after the requisite smearing over invariant mass [10], in terms of

partonic kinematic variables. The exclusive rates on the other hand, are not calculable from first principles and must be parameterized in terms of form factors. Equating the calculable inclusive rate to the sum over exclusive modes leads to the sum rules which will be utilized in this paper. The sum rules are derived by noting that the time-ordered product of two currents between B mesons with four-velocity v ,

$$\begin{aligned} T^{\mu\nu}(v \cdot q, q^2) &= -i \int d^4x e^{-iq \cdot x} \langle B(v) | T \left(J^{\mu\dagger}(x) J^\nu(0) \right) | B(v) \rangle \\ &\equiv -g^{\mu\nu} T_1 + v^\mu v^\nu T_2 - i\epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta T_3 + q^\mu q^\nu T_4 \\ &\quad + (q^\mu v^\nu + q^\nu v^\mu) T_5, \end{aligned} \quad (1)$$

can be expressed as either a sum over hadronic or partonic intermediate states. The former expression contains the matrix elements $\langle H | J | B \rangle$ of interest, while the latter may be expanded [1] as an operator product expansion (OPE) in the heavy quark effective theory[11]. Both the hadronic and OPE-based expressions for the time-ordered product $T^{\mu\nu}$ may be analytically continued to complex $v \cdot q$, holding the three-momentum $q_3 = |\vec{q}|$ fixed. In terms of the variable

$$\epsilon = M_B - E_H - v \cdot q, \quad (2)$$

where $E_H = \sqrt{M_H^2 + q_3^2}$ is the H meson energy and M_B is the B meson mass, $T^{\mu\nu}$ has two branch cuts along the real epsilon axis: a “local” cut for $\epsilon \geq 0$ and a “distant” cut for $\epsilon \leq -2E_H$. Far from these cuts, the OPE-based expression $T_{\mu\nu}^{OPE}$ should reliably approximate the hadronic one.

Contracting with an arbitrary four-vector a^μ and equating the hadronic sum over states to the OPE-based calculation gives

$$\begin{aligned} &\frac{|\langle H | a \cdot J | B \rangle|^2}{4M_B E_H \epsilon} + \sum_{X \neq H} \frac{|\langle X | a \cdot J | B \rangle|^2}{4M_B E_X (\epsilon + E_H - E_X)} \\ &- \sum_X (2\pi)^3 \delta^{(3)}(\vec{p}_X - \vec{q}) \frac{|\langle B | a \cdot J | X \rangle|^2}{4M_B E_X (\epsilon + E_H + E_X - 2M_B)} = a^{\mu*} T_{\mu\nu}^{OPE} a^\nu. \end{aligned} \quad (3)$$

The first two terms represent the local cut, while the third term, which sums over states X containing one q and two b quarks, represents the distant cut. The sum over states contains the usual phase space integration $\int d^3p/(2E)$ for each particle, while $\sum'_{X \neq H}$ is shorthand for

$$\sum'_{X \neq H} \equiv \sum_{X \neq H} (2\pi)^3 \delta^{(3)}(\vec{p}_X + \vec{q}). \quad (4)$$

Eq. 3 is derived by assuming ϵ is real, then analytically continuing to complex ϵ . Following the procedure outlined in references [8] and [9], we integrate in ϵ along a contour that encloses only the local branch cut while remaining far from either cut (except at $\epsilon \rightarrow \infty$, where local duality is expected to work well). The $|\langle B|a \cdot J|X \rangle|^2$ term in Eq. 3 will then give a vanishing contribution. To ensure convergence, we multiply $a^\dagger T a$ by a smooth weight function $W_\Delta(\epsilon)$ satisfying $W_\Delta(0) = 1$, $W_\Delta(\epsilon) \rightarrow 0$ for $\epsilon \ll \Delta$, and $W_\Delta(\epsilon) > 0$ for ϵ real. Δ acts as an ultra-violet cutoff which serves to damp the contribution from excited states. The result of this integration, $\int W_\Delta(\epsilon) d\epsilon$, is the zeroth moment rule

$$\frac{|\langle H|a \cdot J|B \rangle|^2}{4M_B E_H} + \sum_{X \neq H} \frac{|\langle X|a \cdot J|B \rangle|^2}{4M_B E_X} W_\Delta(E_X - E_H) = \int d\epsilon W_\Delta(\epsilon) a^{\mu*} T_{\mu\nu}^{OPE} a^\nu. \quad (5)$$

The positivity of $|\langle X|a \cdot J|B \rangle|^2$ gives an immediate upper bound on the magnitude of the combination of form factors entering $\langle H|a \cdot J|B \rangle$.

Integrating $\int W_\Delta(\epsilon) \epsilon d\epsilon$ gives the sum rule for the first moment,

$$\sum_{X \neq H} \frac{(E_X - E_H) |\langle X|v \cdot J|B \rangle|^2}{4M_B E_X} W_\Delta(E_X - E_H) = \int \epsilon d\epsilon W_\Delta(\epsilon) a^{\mu*} T_{\mu\nu}^{OPE} a^\nu. \quad (6)$$

This leads to lower bounds on form factors by noting that, if E_1 is the energy of the first resonance more massive than H ,

$$(E_1 - E_H) \sum_{X \neq H} \frac{|\langle X|a \cdot J|B \rangle|^2}{4M_B E_X} W_\Delta(E_X - E_H) \leq \sum_{X \neq H} \frac{(E_X - E_H) |\langle X|a \cdot J|B \rangle|^2}{4M_B E_X} W_\Delta(E_X - E_H). \quad (7)$$

We neglect the contribution of multi-particle states with energies less than that of the first excited resonance. The contributions of such states are suppressed by both phase space and large- N_c power counting and moreover, are empirically negligible (e.g., $D \rightarrow K^* \mu \nu$ versus $D \rightarrow K \pi \mu \nu$).

Substituting Eq. 7 into Eq. 6 provides an upper bound on the contribution of excited states to the zeroth moment rule Eq. 5. This in turn implies a lower bound on the hadronic matrix element $\langle H|a \cdot J|B \rangle$. We therefore have both the upper and lower bounds

$$\int d\epsilon W_\Delta(\epsilon) a^{\mu*} T_{\mu\nu}^{OPE} a^\nu \geq \frac{|\langle H|a \cdot J|B \rangle|^2}{4M_B E_H} \geq \int d\epsilon W_\Delta(\epsilon) a^{\mu*} T_{\mu\nu}^{OPE} a^\nu \left[1 - \frac{\epsilon}{E_1 - E_H} \right]. \quad (8)$$

Eq. 7 was previously used for deriving the Voloshin bound on the slope of the $\bar{B} \rightarrow D^* l \bar{\nu}$ form factor at zero recoil[12]. The bounds derived here apply to the normalizations of form factors, rather than the slopes, and may be used away from zero recoil as well. We may now

use Eq. 8 to bound the form factor of our choosing by appropriately selecting the four-vector a_μ and current J_μ . Furthermore, variation of q_3 leads to constraints over the entire physical range of momentum transfer q^2 . When the first moment of $a^\dagger T a$ is small, the upper and lower bounds are close to each other, and the form factor is tightly constrained. Naturally, this is the most interesting kinematic region to consider, but care is required since higher order terms become important.

There are several expansion parameters implicit in Eq. 8. The OPE result contains powers of Λ/m_b from matching to the heavy quark effective theory, $\Lambda/2E_q$ from expressing the time-ordered product as a sum of local operators and Λ/Δ from derivatives of the weight function W_Δ , where Λ is a typical hadronic energy scale. To the order at which we work, the Λ/Δ terms can be eliminated by taking $\Delta \sim E_q$ and choosing a weight function whose first and second derivatives vanish at zero. Thus, $\Lambda/2E_q$ is the limiting parameter and the bounds are only valid for sufficiently large energies, at least $E_q \gtrsim 1 \text{ GeV}$, corresponding to small q^2 . For $\overline{B} \rightarrow \rho, \pi, \omega$, the maximum energy of the final hadron is about 2.7 GeV, so the bounds can be valid over a substantial kinematic range, roughly given by $0 \leq q^2 \lesssim 18 \text{ GeV}^2$. Since our integration contour necessarily approaches either the local or distant branch cut to within E_H , this requirement also enforces the local duality condition that the contour remain far from any cuts.

In addition there are perturbative corrections that we expect, for $\Delta \sim E_q$, to be the same order as the $1/2E_q$ corrections. Schematically, the corrections to the first moment enter in the form

$$\Lambda + \frac{\lambda_1 + \lambda_2}{2E_q} + \frac{\alpha_s(\Delta)}{\pi} \Delta + \dots, \quad (9)$$

where functions of q_3 and particle masses multiply each of the terms above. When the leading Λ term vanishes, both the $1/2E_q$ terms presented in this paper and the uncalculated α_s corrections are dominant. The α_s corrections need to be calculated before our lower bounds can be reliably applied in this kinematic region.

3 The Hadronic Side

To apply the generic bounds Eq. 8 to a specific form factor, we must choose an appropriate current J and four-vector a^μ . The matrix elements for semi-leptonic decay of a B meson

into a pseudoscalar meson P or a vector meson V may be parameterized as

$$\langle P(p') | V^\mu | \bar{B}(p) \rangle = (p + p')^\mu f_+ + (p - p')^\mu f_-, \quad (10)$$

$$\langle V(p') | V^\mu | \bar{B}(p) \rangle = ig\epsilon^{\mu\nu\alpha\beta}\epsilon_\nu^* p'_\alpha p_\beta, \quad (11)$$

$$\langle V(p') | A^\mu | \bar{B}(p) \rangle = f\epsilon^{*\mu} + \left[(p + p')^\mu a_+ + (p - p')^\mu a_- \right] p \cdot \epsilon^*. \quad (12)$$

The states in Eq. (12) have the usual relativistic normalization of $2E$. Contributions to decay rates from a_- and f_- are suppressed by the lepton mass and are therefore of less interest.

The tensor coefficients T_i of the time-ordered product $T^{\mu\nu}$ receive contributions[13] from the above matrix elements. Decays to pseudoscalar mesons contribute

$$\begin{aligned} T_1 &= 0, & T_2 &= 2f_+^2 M_B \frac{1}{\Delta_P}, & T_3 &= 0, \\ T_4 &= (f_+ - f_-)^2 \frac{1}{2M_B \Delta_P}, & T_5 &= f_+ (f_- - f_+) \frac{1}{\Delta_P}, \end{aligned} \quad (13)$$

while decays to vectors contribute

$$\begin{aligned} T_1 &= \left[g^2 (p \cdot q^2 - M_B^2 q^2) + f^2 \right] \frac{1}{2M_B \Delta_V}, \\ T_2 &= \left[-q^2 g^2 + \frac{f^2}{M_V^2} + 4a_+^2 \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_V^2} \right) + 4f a_+ \left(-1 + \frac{M_B^2}{M_V^2} - \frac{p \cdot q}{M_V^2} \right) \right] M_B \frac{1}{2\Delta_V}, \\ T_3 &= g f \frac{1}{\Delta_V}, \\ T_4 &= \left[-g^2 M_B^2 + \frac{f^2}{M_V^2} + (a_+ - a_-)^2 \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_V^2} \right) \right. \\ &\quad \left. + 2f (a_+ - a_-) \frac{M_B^2 - p \cdot q}{M_V^2} \right] \frac{1}{2M_B \Delta_V}, \\ T_5 &= \left[p \cdot q g^2 - \frac{f^2}{M_V^2} + 2a_+^2 \left(M_B^2 - \frac{(M_B^2 - p \cdot q)^2}{M_V^2} \right) + f a_+ \left(1 - 3 \frac{(M_B^2 - p \cdot q)}{M_V^2} \right) \right. \\ &\quad \left. + f a_- \left(\frac{(M_B^2 - p \cdot q)}{M_V^2} - 1 \right) + 2a_+ a_- \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_V^2} \right) \right] \frac{1}{2\Delta_V}. \end{aligned} \quad (14)$$

Δ_H (for $H = P$ or $H = V$) is the H meson inverse propagator defined by $\Delta_H = (p - q)^2 - M_H^2 = \epsilon(\epsilon + 2E_H)$. The contributions from decays to scalar or axial vector mesons are exactly analogous to that of pseudoscalar or vector mesons, respectively, after interchanging vector and axial vector currents $V^\mu \leftrightarrow A^\mu$.

Eq.s 13 and 14 allow us to express the hadronic side of the sum rule, involving $\langle H|a \cdot J|B \rangle$, in terms of the form factors in Eq. 12. Isolating an individual form factor is now reduced to making the appropriate choice for a_μ . It is convenient to go to the B rest frame with the z axis in the direction of \vec{q} , $q = (v \cdot q, 0, 0, q_3)$. In this frame we may isolate the form factor f_- (g) by making the choice $a = (0, 1, 0, 0)$ and $J = V$ (A), which selects the sum rule $T_1^{hadronic} = T_1^{OPE}$. Since decays to scalars do not contribute to T_1 , the first excited resonance has spin/parity $J^P = 1^+$. These resonances are $b_1(1235)$, $K_1(1270)$ and $D_1(2420)$ for the transitions $\overline{B}^0 \rightarrow \rho^+ l^- \overline{\nu}$, $\overline{B} \rightarrow K^* \bar{l}$ and $\overline{B} \rightarrow D^* l \overline{\nu}$ respectively.

Similarly, we may isolate the form factor f_+ via the choice $a = (q_3, 0, 0, v \cdot q)$ and $J = V$, leading to the combination $a^\dagger T a = q^2 T_1 + q_3^2 T_2$. This combination has the advantage that no $J^P = 1^-$ states contribute, so the first excited resonance is again a $J^P = 1^+$ state. Had we instead chosen $a = (q_3, 0, 0, M_B - E_\pi)$ to isolate f_+ , the first excited resonance would have been the ρ , resulting in less stringent bounds.

Isolation of the form factor a_+ requires the use of the heavy quark relation $a_- = -a_+(1 + \mathcal{O}(1/m_b))$. Since $1/m_b$ corrections are smaller than $1/E$ corrections, we can use this relation to eliminate a_- from $a^\dagger T a$. Choosing $a = (E_H, 0, 0, -q_3)$ and $J = A$ then selects $a^\dagger T a = -M_H^2 T_1 + E_H^2 T_2 + (M_B E_H - \epsilon E_H - M_H^2)^2 T_4 + 2E_H (M_B E_H - \epsilon E_H - M_H^2) T_5$, which isolates a_+ . The first excited resonance in this case can be a scalar $J^P = 0^+$ or an axial vector $J^P = 1^+$. These states correspond to $a_0(980)$, $K_1(1270)$ and $D_1(2420)$, for the transitions $\overline{B}^0 \rightarrow \rho^+ l^- \overline{\nu}$, $\overline{B} \rightarrow K^* \bar{l}$ and $\overline{B} \rightarrow D^* l \overline{\nu}$ respectively.

We may also consider the phenomenologically interesting decay $B^- \rightarrow \omega l \overline{\nu}$ by noting that both intermediate ρ^0 and ω states contribute to Eq. 1 when the external state is a charged B^- meson, but only ρ^+ contributes when the external state is neutral. By using isospin to relate $\overline{B}^0 \rightarrow \rho^+$ and $B^- \rightarrow \rho^0$ form factors, we can substitute the upper and lower bounds on $\overline{B}^0 \rightarrow \rho^+ l^- \overline{\nu}$ form factors, Eq. 8, into the sum rules involving ρ^0 and ω intermediate states. This results in upper and lower bounds on form factors for $B^- \rightarrow \omega l \overline{\nu}$.

4 The OPE Side

Having fixed J and a to determine the hadronic side of the sum rule, we need to compute the OPE expression for $a^\dagger T a$. The zeroth order OPE result is simply the naive parton model,

while the leading nonperturbative corrections can be written in terms of the parameters [2-5]

$$\begin{aligned}\lambda_1 &= \langle B(v) | \bar{b}_v (iD)^2 b_v | B(v) \rangle, \\ \lambda_2 &= -Z_b \langle B(v) | \bar{b}_v \frac{g\sigma^{\mu\nu} G_{\mu\nu}}{6} b_v | B(v) \rangle,\end{aligned}\quad (15)$$

where Z_b is a renormalization factor equal to unity at a scale $\mu = m_b$. The bottom quark mass may be eliminated by using the relation

$$m_b = M_B - \bar{\Lambda} + (\lambda_1 + 3\lambda_2)/(2M_B) + \dots \quad (16)$$

The matrix element $\lambda_2 = 0.12 \text{ GeV}^2$ is determined by the $B^* - B$ mass splitting, while λ_1 and $\bar{\Lambda}$ may be extracted from inclusive decay distributions[14, 15].

The zeroth, first, and second moments of T_1, T_2 , and T_3 have been calculated in reference [8]. We also need the moments of T_4 and T_5 for a $b \rightarrow q$ changing axial current (the result for a vector current may be obtained by making the replacing the final quark mass $m_q \rightarrow -m_q$). Due to the mismatch between the definition of ϵ in terms of hadronic variables and the computation of the OPE in terms of partonic variables, the OPE is an expansion about $\delta \equiv E_q - E_H + M_B - m_b$,

$$T_i = \sum_n \frac{A_i^{(n)}}{(\epsilon - \delta)^{n+1}}. \quad (17)$$

The $A_i^{(n)}$ would be the n^{th} moments of T_i if we defined as $\epsilon = m_b - E_q - v \cdot q$. For T_4 , they are given by

$$A_4^{(0)} = \frac{\lambda_1 + 3\lambda_2}{3M_B E_q^3}, \quad A_4^{(1)} = -\frac{\lambda_1 + 3\lambda_2}{3M_B E_q^2}, \quad A_4^{(2)} = 0, \quad (18)$$

while for T_5 , we have

$$\begin{aligned}A_5^{(0)} &= -1/2E_q - \frac{5\lambda_2}{4E_q^3} - \left(\frac{1}{2E_q^3} + \frac{m_q^2}{4E_q^5}\right)\lambda_1, \\ A_5^{(1)} &= \left(\frac{5}{4E_q^2} - \frac{5}{4M_B E_q}\right)\lambda_2 + \left(\frac{1}{2E_q^2} + \frac{m_q^2}{4E_q^4} - \frac{5}{12M_B E_q}\right)\lambda_1, \\ A_5^{(2)} &= \left(\frac{1}{6E_q} - \frac{m_q^2}{6E_q^3}\right)\lambda_1.\end{aligned}\quad (19)$$

Higher moments will not contribute at this order. It is a simple matter to construct the moments of T_i from A_i . The first moment, for example, is $\int \epsilon d\epsilon T = \delta A^{(0)} + A^{(1)}$. For $\bar{B} \rightarrow$

$Dl\bar{\nu}$, δ may be set to zero when multiplying higher order corrections λ_1, λ_2 as $\delta \approx \bar{\Lambda}(w-1)/w$, where w is the velocity transfer $v \cdot v'$. For $\bar{B} \rightarrow \rho l \bar{\nu}$, we keep factors of $E_q - E_H$ multiplying such terms even though $M_H \sim \Lambda$ formally implies $E_q - E_H \sim \mathcal{O}(\Lambda^2/E_q)$, because they are numerically important $M_H^2 \gg \lambda_1, \lambda_2$.

Given a four-vector a^μ , we can now construct OPE expressions for the moments of the sum rule combinations $a^\dagger T a$. Using Eq. 8 then leads to the bounds

$$\begin{aligned}
& \frac{E_q + m_q}{2E_q} + \left(\frac{1}{M_B^2} + \frac{2m_q}{3M_B E_q^2} + \frac{m_q^2}{E_q^4} \right) \frac{m_q \lambda_1}{4E_q} + \left(\frac{3}{M_B^2} + \frac{2m_q}{M_B E_q^2} - \frac{1}{E_q^2} \right) \frac{m_q \lambda_2}{4E_q} \\
& \geq \frac{f^2}{4M_B E_H} \\
& \geq \frac{1}{2(E_1 - E_H)} \left[\frac{E_q + m_q}{E_q} (E_1 - E_q - \bar{\Lambda}) \right. \\
& \quad + \left(\frac{1}{3E_q} - \frac{1}{3M_B} - \frac{m_q}{2M_B^2} + \frac{m_q E_1}{2M_B^2 E_q} + \frac{m_q^2}{6E_q^3} + \frac{m_q^2 E_1}{3M_B E_q^3} + \frac{m_q^3 E_1}{2E_q^5} \right) \lambda_1 \\
& \quad \left. + \left(-\frac{1}{2E_q} - \frac{1}{M_B} - \frac{3m_q}{2M_B^2} - \frac{m_q E_1}{2E_q^3} + \frac{3m_q E_1}{2M_B^2 E_q} + \frac{m_q^2 E_1}{M_B E_q^3} \right) \lambda_2 \right] \quad (20)
\end{aligned}$$

for the final states $H = D^*, K^*$, or ρ^+ . The upper and lower bounds on $f^2/(4M_B E_H)$ in the above equation serve also to bound the vector form factor $g^2 M_B q_3^2/(4E_H)$ after replacing $m_q \rightarrow -m_q$.

The bounds on a_+ , which involve higher moments, are given by

$$\begin{aligned}
& \frac{E_H^2 + q_3^2}{2} - \frac{E_H q_3^2}{E_q} - \frac{m_q M_H^2}{2E_q} \\
& + \left[\frac{q_3^4}{3M_B E_q^3} - \frac{E_H q_3^2}{3E_q^3} - \frac{E_H^2}{3E_q M_B} - \frac{M_H^2 m_q}{4M_B^2 E_q} - \left(\frac{M_H^2}{6M_B E_q^3} + \frac{E_H q_3^2}{2E_q^5} \right) m_q^2 - \frac{M_H^2 m_q^3}{4E_q^5} \right] \lambda_1 \\
& + \left[\frac{q_3^4}{M_B E_q^3} - \frac{E_H q_3^2}{2E_q^3} - \frac{E_H^2}{M_B E_q} - \left(\frac{3E_H^2 + q_3^2}{4E_q^3} + \frac{3M_H^2}{4M_B^2 E_q} \right) m_q - \frac{M_H^2 m_q^2}{2M_B E_q^3} \right] \lambda_2 \\
& \geq \frac{q_3^2 M_B M_H^2 a_+^2}{E_H} \\
& \geq \frac{1}{E_1 - E_H} \left\{ \left[\frac{E_H^2 + q_3^2}{2} - \frac{E_H q_3^2}{E_q} - \frac{m_q M_H^2}{2E_q} \right] (E_1 - E_q - \bar{\Lambda}) \right. \\
& \quad + \left[\frac{E_H^2 + q_3^2}{6E_q} - \frac{E_H q_3^2 + E_1 E_H^2}{3E_q M_B} + \frac{2E_H^2 + M_H^2}{6M_B} - \frac{E_H q_3^2 E_1}{3E_q^3} + \frac{q_3^4 E_1}{3M_B E_q^3} \right. \\
& \quad + \frac{M_H^2 m_q}{4M_B^2} \left(1 - \frac{E_1}{E_q} \right) - \left(\frac{E_1 M_H^2}{6M_B E_q^3} - \frac{E_H^2 + q_3^2}{12E_q^3} + \frac{E_H E_1 q_3^2}{2E_q^5} \right) m_q^2 - \frac{E_1 M_H^2 m_q^3}{4E_q^5} \left. \right] \lambda_1 \\
& \quad + \left[\frac{2E_H^2 + M_H^2}{2M_B} + \frac{2E_H^2 + M_H^2}{4E_q} - \frac{E_1 E_H^2 + E_H q_3^2}{E_q M_B} - \frac{E_H E_1 q_3^2}{2E_q^3} + \frac{E_1 q_3^4}{M_B E_q^3} \right.
\end{aligned}$$

$$- \left(\frac{E_1(3E_H^2 + q_3^2)}{4E_q^3} - \frac{3M_H^2}{4M_B^2} + \frac{3E_1M_H^2}{4M_B^2E_q} \right) m_q - \frac{E_1M_H^2m_q^2}{2M_BE_q^3} \lambda_2 \Big\}. \quad (21)$$

For the sake of bookkeeping we have retained the $1/M_B$ terms even though the relation $a_- = -a_+$ used on the hadronic side of this sum rule is valid only to $\mathcal{O}(M_B^0)$. When $m_q = 0$, the zeroth order term in the upper bound, $(E_\rho - q_3)^2$, should naively be $\mathcal{O}(q_3^2)$, but is actually $\mathcal{O}(\Lambda^4/q_3^2)$ for $q_3 \gg m_\rho \sim \Lambda$, so the λ_1, λ_2 terms are formally leading. Thus, a_+ is suppressed, relative to naive expectations, by either $1/2E_q$ or α_s .

Bounds on f_+ for $\bar{B}^0 \rightarrow \pi^+ l^- \bar{\nu}$ also involve higher moments since the four-vector $a = (q_3, 0, 0, v \cdot q)$ depends on ϵ . We find

$$\begin{aligned} & \frac{E_q - m_q}{2E_q} (M_B - \bar{\Lambda} + m_q)^2 + \left[\frac{M_B}{E_q} - \frac{5}{6} + \left(\frac{1}{2M_B} - \frac{1}{12E_q} \right) m_q + \frac{M_B}{6E_q^3} m_q^2 \right. \\ & + \left(\frac{-M_B^2}{4E_q^5} + \frac{1}{3E_q^3} - \frac{1}{4M_B^2E_q} \right) m_q^3 + \left(\frac{-M_B}{2E_q^5} + \frac{1}{6M_BE_q^3} \right) m_q^4 - \frac{1}{4E_q^5} m_q^5 \Big] \lambda_1 \\ & + \left[-\frac{1}{2} + \frac{M_B}{E_q} + \left(\frac{3}{2M_B} + \frac{M_B^2}{4E_q^3} - \frac{1}{4E_q} \right) m_q - \frac{M_B}{E_q^3} m_q^2 - 3 \left(\frac{1}{4E_q^3} + \frac{1}{4M_B^2E_q} \right) m_q^3 + \frac{m_q^4}{2M_BE_q^3} \right] \lambda_2 \\ & \geq \frac{f_+^2 M_B q_3^2}{E_\pi} \\ & \geq \frac{1}{(E_1 - E_H)} \left\{ \frac{(E_q - m_q)}{2E_q} (M_B - \bar{\Lambda} + m_q)^2 (E_1 - E_q - \bar{\Lambda}) \right. \\ & + \left[5E_q - 5E_1 - 7M_B + \frac{6M_BE_1 + M_B^2}{E_q} + \left(-\frac{3}{2} + \frac{3E_1 - 3E_q}{M_B} + \frac{4M_B - E_1}{2E_q} \right) m_q \right. \\ & + \left(\frac{2M_BE_1 + M_B^2}{2E_q^3} - \frac{1}{M_B} + \frac{1}{E_q} \right) m_q^2 + \left(\frac{M_B + 2E_1}{E_q^3} + \frac{3E_q - 3E_1}{2M_B^2E_q} - \frac{3M_B^2E_1}{2E_q^5} \right) m_q^3 \\ & + \left(\frac{M_B + 2E_1}{2M_BE_q^3} - \frac{3M_BE_1}{E_q^5} \right) m_q^4 - \frac{3E_1}{2E_q^5} m_q^5 \Big] \frac{\lambda_1}{6} \\ & + \left[2(M_B + E_q - E_1) + \frac{4M_BE_1 - M_B^2}{E_q} + \left(5 + 6 \frac{E_1 - E_q}{M_B} - \frac{2M_B + E_1}{E_q} + \frac{M_B^2E_1}{E_q^3} \right) m_q \right. \\ & \left. - \left(\frac{2}{M_B} + \frac{1}{E_q} + \frac{4M_BE_1}{E_q^3} \right) m_q^2 + 3 \left(\frac{1}{M_B^2} - \frac{E_1}{E_q^3} - \frac{E_1}{M_B^2E_q} \right) m_q^3 + \frac{2E_1}{M_BE_q^3} m_q^4 \right] \frac{\lambda_2}{4} \Big\}. \quad (22) \end{aligned}$$

For $B^- \rightarrow \omega l \bar{\nu}$, we present bounds only for the form factor $f^{(B-\omega)}$. Since these are derived by combining two sum rules, they depend on the energy E_1 of the first neutral $J^P = 1^-$ resonance above the ω , the $\Phi(1020)$, as well as the energy E_{b1} of the first charged $J^P = 1^+$ resonance above the ρ , the $b_1(1235)$. Setting $m_q = 0$ and $E_q = q_3$ gives the bounds

$$\frac{1}{4(E_{b1} - E_\rho)} \left[E_{b1} + q_3 - 2E_\rho + \bar{\Lambda} + \left(\frac{1}{3M_B} - \frac{1}{3q_3} \right) \lambda_1 + \left(\frac{1}{M_B} + \frac{1}{2q_3} \right) \lambda_2 \right]$$

$$\begin{aligned}
&\geq \frac{f_{(B-\omega)}^2}{4M_B E_\omega} \\
&\geq \frac{1}{4(E_1 - E_\omega)} \left[E_1 + E_\rho - 2q_3 - 2\bar{\Lambda} + \left(\frac{-2}{3M_B} + \frac{2}{3q_3} \right) \lambda_1 - \left(\frac{2}{M_B} + \frac{1}{q_3} \right) \lambda_2 \right] \quad (23)
\end{aligned}$$

The upper bound on $f^{(B-\omega)}$ is better than the naive one in Eq. 8 for larger values of q^2 , roughly when $q_3 < (M_{b1}^2 - \bar{\Lambda}^2)/(2\bar{\Lambda})$. The lower bound is only useful for large momentum transfer and rather small values of $\bar{\Lambda}$.

5 Discussion

Let us consider the reliability of the bounds derived above. We notice that the upper bounds rely only upon the zeroth moments and generally receive small corrections from $1/E$ nonperturbative terms. Perturbative α_s corrections should be similarly small, so most of the upper bounds are trustworthy. An exception is the upper bound on a_+ . In this case the zeroth moment is dominated by both the $1/2E_q$ and α_s terms, and our result must be supplemented by a perturbative calculation. Even without such a calculation, we see that a_+ is dynamically suppressed for large E_ρ , *i.e.* for small momentum transfer $q^2 \lesssim 18 \text{ GeV}^2$. There have been attempts to calculate form factors such as a_+ within the confines of perturbative QCD utilizing Sudakov resummations to avoid the use of arbitrary cutoffs in the end point region, see [16]. These methods also predict an α_s suppression for small q^2 , but their normalization depends on unknown hadronic wavefunctions.

The lower bounds exhibit cancelations over much of the q^2 range and are most interesting when the first moment is small, of order the $1/2E_q$ terms, thus making the inclusion of short distance corrections imperative. Nevertheless, the $\mathcal{O}(\alpha_s^0)$ formulas presented here are a necessary first step and give a rough idea of how constraining the lower bounds might be. Since the precise numerics are irrelevant without the α_s corrections, we will only discuss the qualitative behavior of some representative bounds.

Plotted in Fig. 1a are upper and lower bounds on the $\bar{B}^0 \rightarrow \rho^+ l^- \bar{\nu}$ form factor $f/(M_B + M_\rho)$ as a function of momentum transfer q^2 . The lower bound, displayed for a range of correlated $\bar{\Lambda}, \lambda_1$ values taken from reference [14], depends sensitively on the values of $\bar{\Lambda}, \lambda_1$, and λ_2 , while the upper bound (solid line) has no dependence on them at all. The dashed line is the lower bound without higher order corrections, using $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = 0, \lambda_2 = 0$. We see that the bounds cross at $q^2 \sim 18 \text{ GeV}^2$, indicating the need for higher order corrections. The results

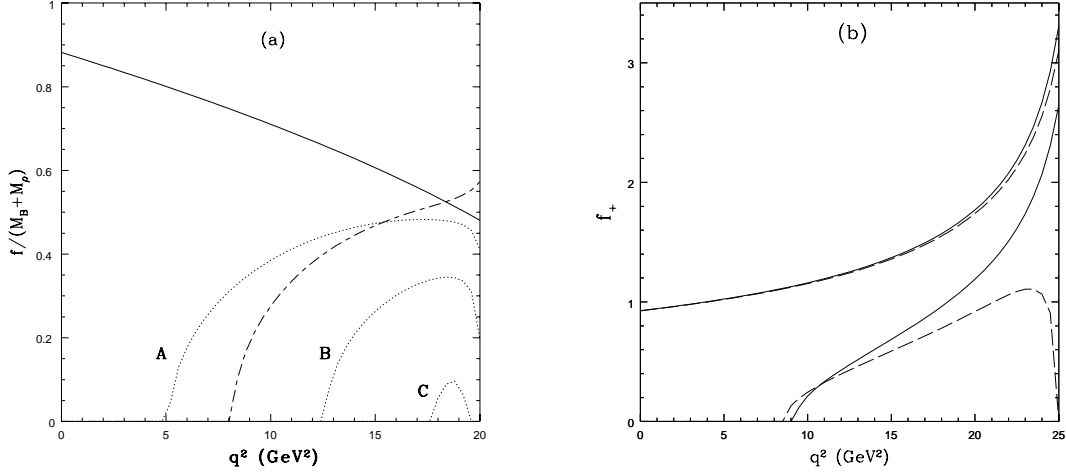


Figure 1: *Upper and lower bounds on :* (a) The $\overline{B}^0 \rightarrow \rho^+ l^- \overline{\nu}$ form factor $f(q^2)/(M_B + M_\rho)$. Solid and dashed lines are bounds for $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = \lambda_2 = 0$; Dotted lines correspond to values given in the text. (b) The $\overline{B}^0 \rightarrow \pi^+ l^- \overline{\nu}$ form factor $f_+(q^2)$. Solid lines are for $\bar{\Lambda} = 0.39, \lambda_1 = \lambda_2 = 0$, dashed lines $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = -0.19 \text{ GeV}^2, \lambda_2 = 0.12 \text{ GeV}^2$.

of including the $1/E$ corrections, using the measured value $\lambda_2 = 0.12 \text{ GeV}^2$, are illustrated by the dotted lines A, B, C choosing the values, $[\bar{\Lambda} = 0.28 \text{ GeV}, \lambda_1 = -0.09 \text{ GeV}^2]$, the central values [9] $[\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = -0.19 \text{ GeV}^2]$ and $[\bar{\Lambda} = 0.50 \text{ GeV}, \lambda_1 = -0.29 \text{ GeV}^2]$ respectively. Clearly, the bounds are much more restrictive for low values of $\bar{\Lambda}$ and $|\lambda_1|$. All the bounds except (C) are in a range relevant to ruling out typical models.

In Fig. 1b the upper and lower bounds on the $\overline{B}^0 \rightarrow \pi^+ l^- \overline{\nu}$ form factor f_+ are plotted in solid lines for $\bar{\Lambda} = 0.39$ with vanishing λ_1 and λ_2 and in dashed lines for $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = -0.19 \text{ GeV}^2, \lambda_2 = 0.12 \text{ GeV}^2$. As with $\overline{B}^0 \rightarrow \rho^+ l^- \overline{\nu}$ the bounds are more restrictive for low values of $\bar{\Lambda}$ and $|\lambda_1|$. For example, the model of Wirbel, Stech, and Bauer[17] is barely compatible with the lower dashed bound and is incompatible if $\bar{\Lambda} = 0.28 \text{ GeV}^2$ is used instead (although no conclusions about the reliability of such models can be made without the α_s corrections).

Any bound with $m_q = 0$ becomes unreliable when E_H is too small, *i.e.* when q^2 is too large. A hadronic energy greater than $\sim 1 \text{ GeV}$, corresponding to $q^2 \lesssim 18 \text{ GeV}^2$ for $\overline{B} \rightarrow \rho l \overline{\nu}$ or $\overline{B} \rightarrow \pi l \overline{\nu}$, is probably necessary for the $1/2E_q$, α_s , and local duality corrections to be under control. When $m_q = m_c$, on the other hand, such corrections can be under control even at zero recoil, $q_3 = 0$. The bounds on the $\overline{B} \rightarrow D^* l \overline{\nu}$ form factor $f/[\sqrt{M_B M_{D^*}}(1+w)]$,

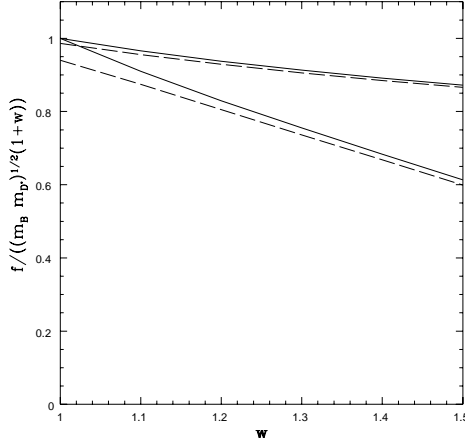


Figure 2: Upper and lower bounds on the $\overline{B} \rightarrow D^* l \overline{\nu}$ form factor $f/[(M_B M_{D^*})^{1/2}(1+w)]$, normalized to coincide with the Isgur-Wise function in the heavy quark limit. Solid lines are for $\bar{\Lambda} = 0.39, \lambda_1 = 0, \lambda_2 = 0$, dashed lines for $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = -0.19 \text{ GeV}^2, \lambda_2 = 0.12 \text{ GeV}^2$.

normalized to coincide with the Isgur-Wise function in the infinite mass limit, are plotted against velocity transfer $w = v \cdot v'$ in Fig. 3. Upper and lower bounds are shown in solid lines for the leading order result $\bar{\Lambda} = 0.39, \lambda_1 = 0, \lambda_2 = 0$ and in dashed lines for $\bar{\Lambda} = 0.39 \text{ GeV}, \lambda_1 = -0.19 \text{ GeV}^2, \lambda_2 = 0.12 \text{ GeV}^2$. Both sets of bounds are easily compatible with ALEPH[18] data but only marginally compatible with CLEO[19] and DELPHI[20] data. Differentiating the upper and lower bounds at zero recoil with respect to w leads to the generalizations of the Bjorken[21] and Voloshin[12] inequalities on the slope of the Isgur-Wise function. Perhaps even more interesting are the bounds on the normalization at zero recoil.

The α_s and part of the α_s^2 corrections to the upper bound have been computed[2, 9, 22] but indicate the possibility of poor convergence for small Δ . At order α_s , the difference between the upper and lower bounds on $f(1)/[2\sqrt{M_B M_{D^*}}]$ is $\frac{-\alpha_s(\Delta)}{\pi(M_1 - M_{D^*})} X_A^{(1)}$, where

$$\begin{aligned}
X_A^{(1)} &= \frac{m_c(m_c + M_B)(m_c - 3M_B)}{9M_B^2} \ln \frac{\Delta + m_c}{m_c} \\
&- \frac{\Delta}{54M_B^2(\Delta + m_c)^2} \left[6m_c^4 - 12m_c M_B \Delta^2 - 18m_c^2 M_B \Delta - 12m_c^3 M_B + 9m_c^3 \Delta \right. \\
&- \left. 14\Delta^3 m_c - 10m_c^2 \Delta^2 - 27M_B^2 \Delta m_c - 18M_B^2 \Delta^2 - 4\Delta^4 - 18M_B^2 m_c^2 \right]. \quad (24)
\end{aligned}$$

Using $\bar{\Lambda} = 0.39 \text{ GeV}$, $\lambda_1 = -0.19 \text{ GeV}^2$, $\lambda_2 = 0.12 \text{ GeV}^2$ and a weight function $W_\Delta = \theta(\Delta - \epsilon)$ with $\Delta = 1 \text{ GeV}$ and $\alpha_s(\Delta) = 0.45$, we find that α_s corrections move the upper and lower bounds at threshold from $0.99 \geq f(1)/[2\sqrt{M_B M_{D^*}}] \geq 0.94$ to $0.96 \geq f(1)/[2\sqrt{M_B M_{D^*}}] \geq 0.90$. The upper bound remains 0.96 for higher Δ values, $\Delta = 2 \text{ GeV}$ and $\alpha_s(\Delta) = 0.28$ or $\Delta = 3 \text{ GeV}$ and $\alpha_s(\Delta) = 0.23$, while the lower bound decreases to 0.88 or 0.86 respectively. A better understanding of the convergence properties in α_s is needed here.

6 Conclusions

We have used inclusive sum rules to derive model-independent upper and lower bounds on the form factors f, g, a_+ and f_+ for $\bar{B} \rightarrow D, D^*, \rho, \pi, K$ and K^* semi-leptonic and radiative decays, as well as upper and lower bounds on the $B \rightarrow \omega$ form factor f . The method is easily generalized to other form factors or combinations of form factors and can be systematically improved by retaining higher order corrections in $1/2E_q$ or α_s .

At leading order, we find a surprising suppression of the $\bar{B}^0 \rightarrow \rho^+ l^- \bar{\nu}$ form factor a_+ at small momentum transfer, an experimentally verifiable prediction. Other heavy-to-light form factors have upper and lower bounds that are comparable to typical models. We have included the leading $1/2E_q$ nonperturbative corrections but not the α_s corrections. This is generally sufficient for reliable upper bounds but not for lower bounds, which may be significantly modified by the α_s corrections.

For $\bar{B} \rightarrow D^* l \bar{\nu}$, we expect the α_s corrections to alter the bounds by only a few percent, so these are reliable to that accuracy. We computed the α_s correction to the lower bound on the form factor $f(1)/2\sqrt{M_B M_{D^*}}$ at zero recoil. This widens the gap between the upper and lower bounds by only 0.01 for $\Delta = 1 \text{ GeV}$. This should prove useful for extracting V_{cb} , as long as the α_s^2 corrections can be brought under control.

The phenomenological implications of the sum rule bounds must await a computation of the α_s corrections away from zero recoil. We hope to present such an analysis in a later publication[23]. Once these terms are under control, the sum rule bounds may provide a means to not only rule out various models, but also to constrain the values of the CKM elements V_{cb} , V_{ub} , and V_{ts} from decays like $\bar{B} \rightarrow D, D^*, l \bar{\nu}$, $\bar{B} \rightarrow \rho, \pi, l \bar{\nu}$, and $\bar{B} \rightarrow K^* \gamma$. For example, the $\bar{B}^0 \rightarrow \rho^+ l^- \bar{\nu}$ bounds with $\bar{\Lambda} = 0.39 \text{ GeV}$, $\lambda_1 = 0$, and $\lambda_2 = 0$, evaluated at $q^2 = 12 \text{ GeV}^2$, combined with a model-independent parameterization of f [24], and lattice

results at a single kinematic point $q^2 \sim q_{max}^2$ [25] constrain the total rate for $\overline{B} \rightarrow \rho l \overline{\nu}$ to better than 40%. Eventually, it may be possible to forego lattice simulations in favor of experimental data by using SU(3) and heavy quark symmetries[26].

How good the constraints will be in reality depend crucially on the size and form of the α_s corrections as well as the actual values of $\bar{\Lambda}$ and λ_1 . The former can be addressed by explicit computation while the latter must await better experimental data (e.g., on the differential electron distribution in $\overline{B} \rightarrow X_s \gamma$). The possibility of making model-independent extractions of CKM elements like V_{ub} and V_{ts} is tantalizing and warrants continued investigation in this area.

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References

- [1] J. Chay, H. Georgi and B. Grinstein, Phys. Lett. **B247**, 399 (1990).
- [2] B. Bigi, M. Shifman, N. Uraltsev and A.I. Vainshtein, Phys. Rev. Lett. **71**, 496 (1993).
- [3] B. Blok, L. Koyrakh, M. Shifman and A.I. Vainshtein, Phys. Rev. **D49**, 3356 (1994).
- [4] A.V. Manohar and M.B. Wise, Phys. Rev. **D49**, 1310 (1994).
- [5] A.F. Falk, M.E. Luke and M.J. Savage, Phys. Rev. **D49**, 3367 (1994).
- [6] T. Mannel, Nucl. Phys. **B413**, 396 (1994).
- [7] R. Akhouri and I.Z. Rothstein, hep-ph/9512303, to appear in *Phys. Rev. D*.
- [8] B. Bigi, M. Shifman, N. Uraltsev and A.I. Vainshtein, Phys. Rev. **D52**, 196 (1995).
- [9] B. Grinstein, A. Kapustin, Z. Ligeti and M. B. Wise, CALT-68-2041, hep-ph/9602262.
- [10] E.C. Poggio, H.R. Quinn and S. Weinberg, Phys. Rev. **D13**, 1958 (1976).
- [11] N. Isgur and M. Wise, Phys. Lett. **B232**, 113 (1989); E. Eichten and B. Hill, Phys. Lett. **B234**, 511 (1990); M. B. Voloshin and M. A. Shifman, Yad. Fiz. **47**, 801 (1988) [Sov. J. Nucl. Phys. 47, 511 (1988)].
- [12] M.B. Voloshin, Phys. Rev. **D46**, 3062 (1992).
- [13] C.G. Boyd, B. Grinstein and A.V. Manohar, UCSD/PTH 95-19, to appear in *Phys. Rev. D*.
- [14] B. Grinstein, A. Kapustin, Z. Ligeti and M. B. Wise, CALT-68-2043, hep-ph/9603314.
- [15] A.F. Falk, M.E. Luke and M.J. Savage, Phys. Rev. **D53**, 2491 (1996); Phys. Rev. **D53**, 6316 (1996).
- [16] R. Akhouri and George Sterman, Phys.. Rev. **D50**, 358 (1994).
R. Akhouri and I.Z. Rothstein, Phys. Lett. **B337**, 176 (1994).
M. Dahm, R. Jakob and P. Kroll, Z. Phys. **C68**, 595, (1995).
- [17] M. Bauer, B. Stech and M. Wirbel, Z. Phys. **C34**, 103 (1987).

- [18] D. Buskulic *et al.* (ALEPH Collaboration), Phys. Lett. **B359** 236 (1995).
- [19] B. Barish *et al.* (CLEO Collaboration), Phys. Rev. **D51**, 1014 (1995).
- [20] P. Abreau *et al.* (DELPHI Collaboration), CERN-PPE/96-11.
- [21] J.D. Bjorken, invited talk at Les Rencontres de Physique de la Vallee d'Aoste, La Thuile, Italy, SLAC Report No. SLAC-PUB-5278 (1990), unpublished;
N. Isgur and M. Wise, Phys. Rev. **D43**, 819 (1991);
J.D. Bjorken, I. Dunietz and J. Taron, Nucl. Phys. **B371**, 111 (1992).
- [22] A. Czarnecki, TTP96-05, hep-ph/9603261.
- [23] C.G. Boyd, I. Rothstein, in preparation.
- [24] C.G. Boyd, B. Grinstein and R.F. Lebed, Phys. Rev. Lett. **74**, 4603 (1995); Phys. Lett. **B353**, 306 (1995); Nucl. Phys. **B461**, 493 (1996); C.G. Boyd and R.F. Lebed, UCSD-PTH-95-23, hep-ph/9512363.
- [25] J.M. Flynn *et al.* (UKQCD Coll.), Nucl. Phys. **B461**, 327 (1996).
- [26] Z. Ligeti and M. B. Wise, CALT-68-2029, hep-ph/9512225.